Manipulability of Closed Kinematic Chains

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Abstract

This paper presents a coordinate-invariant differential geometric analysis of manipulability for closed kinematic chains containing active and passive joints. The formulation treats both redundant and nonredundant mechanisms, as well as over-actuated and exactly actuated ones, in a uniform manner. Dynamic characteristics of the mechanism and manipulated object can also be naturally included by an appropriate choice of Riemannian metric. We illustrate the methodology with several closed chain examples, and provide a practical algorithm for manipulability analysis of general chains.

1 Introduction

The notion of manipulability has been particularly useful in the analysis and design of robot manipulators. Manipulability, or the ability to move and apply forces in arbitrary directions, has been used in applications ranging from the kinematic design of robotic fingers (Salisbury and Craig 1982), to the optimal positioning of the workpiece in a robot's workspace (Klein and Blaho 1987). Salisbury and Craig (1982) first proposed the concept in the design context mentioned above, while Yoshikawa (1985) gave one of the first comprehensive mathematical treatments of manipulability for general open chains. Since then alternate formulations and various aspects of open chain manipulability have also been investigated in, *e.g.*, Gosselin (1990), Gosselin and Angeles (1991), Angeles (1992), Klein and Miklos (1991).

Nearly all of the previous work on open chain manipulability analysis begins by parametrizing the forward kinematics in local coordinates, and measuring in some fashion the extent to which the Jacobian of the forward kinematics (*i.e.*, the linear map relating joint rates to tip reference frame rates) preserves distances and angles; Section 2 discusses this idea in more detail. In Park and Brockett (1994) and Park (1995a) these and other manipulability measures are formulated under a general mathematical framework using the coordinate-free methods of Riemannian geometry. While the coordinate-free methods resolve some apparent ambiguities that arise in defining a combined position and orientation manipulability, the geometric analysis more or less confirms the intuitive approach of the coordinate-based methods. This is due in part to the fact that the configuration (joint) space for open chains can be regarded as a vector space. Moreover, since all joints are assumed actuated, a coordinate-based formulation of manipulability is particularly straightforward and easy to visualize in this case.

Closed chains, on the other hand, present a number of subtleties. Kinematically speaking, the most obvious difference is that a closed chain's configuration space will no longer be a flat[1] space; in general it will be a curved multi-dimensional surface embedded in a higher-dimensional (typically flat) space. Also, dual to the open chain case, the forward kinematics for closed chains generally is more difficult to solve than the inverse kinematics, with the possibility of multiple solutions; this potentially presents computational difficulties in manipulability analysis, since the forward kinematics plays a crucial role. Another important difference is that an arbitrary subset of the joints may be actuated, in some cases the number of actuators exceeding the mechanism's degrees of freedom (e.g., two cooperating robots holding a common object). It is not difficult to see that any reasonable notion of manipulability should be influenced by the number and choice of actuated joints.

[1] A Riemannian manifold is said to be *flat* if it is *locally isometric* to $\Re^n$; this is discussed in more detail below.

A number of coordinate-based formulations for closed chain kinematic manipulability have been proposed for parallel manipulators (Gosselin and Angeles 1988, Zanganeh and Angeles 1997) and for cooperating robot systems (Lee 1989, Kokkinis and Paden 1989, Li et al 1989, Chiacchio et al 1991, Bicchi et al 1995). In the former case the number of actuators is identical to the mechanism's kinematic degrees of freedom, while in the latter case all the joints of the mechanism are assumed. Existing methods for manipulability analysis of parallel manipulators are formulated in terms of the mapping between actuated joints and the end-effector frame, not unlike the open chain case. The methods for manipulability analysis of cooperating robot systems are more varied; they involve, among other things, approximating the intersection of the manipulablity ellipsoids of the individual robots (Lee 1989), computing the force and velocity polytopes associated with the individual robots and combining them via geometric operations (Kokkinis and Paden 1989), and computing the eigenvalues and eigenvectors of a quadratic form constructed from the grasp matrix (Li et al 1989) and/or the Jacobians of the individual robots (Chiacchio et al 1991, Bicchi et al 1995).

Unfortunately the methods above do not extend in a natural way to those mechanisms which lie between these two ends of the spectrum. Because of the nonlinear characteristics unique to closed chain mechanisms as discussed above, particular care must be exercised in formulating coordinate-based manipulability measures. Not taking into explicit consideration the underlying geometry of closed chains results in the many ad hoc formulations of manipulability found in the literature, and can even produce seemingly paradoxical results (Melchiorri 1993, Chiacchio et al 1993).

In this paper we present a differential geometric formulation of manipulability for general closed chains with an arbitrary number of actuated joints. Manipulability, like any other performance measure, should be formulated such that it is independent of the choice of coordinates, and takes into account the nonlinearity of both the configuration space and the rigid-body displacements. We argue that the global, coordinate-free methods of differential geometry provide the most natural means of formulating manipulability for mechanisms with curved configuration spaces. In this framework actuator characteristics are reflected in the choice of a (pseudo-)Riemannian metric on the configuration space. Inertial characteristics of the closed chain and the manipulated object can also be included in a natural way under this framework. As we show below, this geometric framework uniformly treats both redundant and nonredundant mechanisms, as well as exactly actuated and over-actuated systems, eliminating the need for ad hoc methods.

The paper is organized as follows. In Section 2 we review the geometric formulation of manipulability for open chain mechanisms as discussed in Park and Brockett (1994), briefly reviewing the essential differential geometric background. In Section 3 this formulation of manipulability is extended to general closed chain systems, while Section 4 presents a practical algorithm along with several closed-chain examples. Section 5 extends the kinematic manipulability framework to include inertial properties of the mechanism and manipulated object, along with an example involving two cooperating six d.o.f. robots. Preliminary results of this research have also been reported in Park and Kim (1996), and the rough idea for our closed chain manipulability formulation is very briefly sketched out near the end of the survey paper (Park 1995a).

2 Manipulability and Riemannian Geometry

In this section we present a Riemannian geometric formulation of manipulablity for general chains, beginning with some of the ideas presented in Park and Brockett (1994) and Park (1995a). For a particularly accessible reference on differential geometry, see Spivak (1979).

One of the first papers to consider manipulability is the work of Salisbury and Craig on articulated hands (1982). At issue is the way in which input joint velocity errors propagate to the output velocities of each fingertip. For a two-link planar open chain this phenomenon can be visualized as follows. Let $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and $\mathbf{x} = (x_1, x_2)$ denote local coordinates for the joint space and tip position, respectively, and $\mathbf{J}(\boldsymbol{\theta})$ the Jacobian of the forward kinematic map; then $\dot{\mathbf{x}} = \mathbf{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$. If we now map the unit circle $\{\dot{\boldsymbol{\theta}} \in \Re^n \,|\, \|\dot{\boldsymbol{\theta}}\| = 1\}$ by $\mathbf{J}$ to the space of Cartesian velocities $\dot{\mathbf{x}}$, then the image is an ellipsoid whose principal axes and lengths are given by the eigenvectors and eigenvalues of $\mathbf{J}\mathbf{J}^{\mathrm{T}}$, respectively. Specifically, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of $\mathbf{J}\mathbf{J}^{\mathrm{T}}$, and their corresponding eigenvalues are respectively λ_1 and λ_2, then $\mathbf{v}_1$ is one principal axis of the ellipsoid of length $\sqrt{\lambda_1}$.

The other principal axis $\mathbf{v}_2$ is of length $\sqrt{\lambda_2}$. This is the *manipulability ellipsoid* as described in Yoshikawa (1985); in an ideal posture the manipulablility ellipsoid would be perfectly spherical and of unit radius, indicating no distortion in the Jacobian. Stated another way, the forward kinematic map locally preserves distances and angles in such a posture, and hence is a (local) *isometry*. For arbitrary open chains, if the ratio $\frac{\lambda_{max}}{\lambda_{min}}$ of the maximum and minimum eigenvalues of $\mathbf{J}\mathbf{J}^{\mathrm{T}}$, or the *condition number*, were 1, then the manipulator is said to be at an *isotropic* posture (see, e.g., Angeles (1992)). An isotropy therefore maps unit spheres in the space of joint rates to spheres of arbitrary radius in the space of end-effector rates; for an isometry the image sphere must also be of unit radius.

If $\boldsymbol{\tau}$ is the vector of joint torques and $\mathcal{F}$ is the vector of external force applied at the tip, then from virtual work considerations $\boldsymbol{\tau} = \mathbf{J}^{\mathrm{T}}\boldsymbol{\mathcal{F}}$. Then assuming the manipulator is not at a singular configuration one can write $\boldsymbol{\mathcal{F}} = \mathbf{J}^{-\mathrm{T}}\boldsymbol{\tau}$. If the mechanism is now regarded as a device whose input torques produce output forces, then the *force manipulability* ellipsoid is determined by the eigenvectors and eigenvalues of $(\mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1}$; its eigenvectors will still be $\mathbf{v}_1$ and $\mathbf{v}_2$, but their corresponding eigenvalues will now be $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$. While the standard manipulability ellipsoid reflects the uniformity in the mechanism's velocity gain, the force manipulability ellipsoid reflects the uniformity in the force-torque gain. From the above simple calculation it becomes clear why velocity and force manipulability are said to be dual to each other. In the case of redundant mechanisms there are a few additional subtleties that we shall address later.

With this intuitive explanation of manipulability we now illustrate its geometric formulation via another simple example, this time involving the *2R* spherical open chain of Figure 1.

Consider the *2R* spherical mechanism of Figure 1. If we concern ourselves only with the Cartesian position of the tip and ignore orientation, the forward kinematics of this mechanism can then be regarded as a mapping from the two-dimensional torus, T^2, to the unit two-sphere, S^2. T^2 can be coordinatized by the square $[0, 2\pi] \times [0, 2\pi]$ in $\Re^2$, with local coordinates u^1, u^2. S^2 can be coordinatized using spherical coordinates f^1, f^2: if (F^1, F^2, F^3) are coordinates for the embedding space $\Re^3$, then $F^1 = \cos f^1 \sin f^2$,

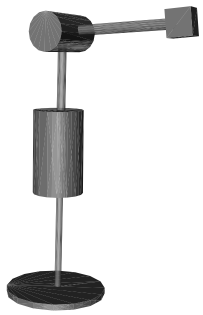

Figure 1: *2R* spherical mechanism.

$F^2 = \sin f^1 \sin f^2$, and $F^3 = \cos f^2$. The workspace of this mechanism is the entire sphere S^2, and its volume is given by the surface area of the sphere, 4π. It is instructive to review the derivation of this number. From the Pythagorean Theorem, the incremental arc length in Euclidean 3-space is given by $ds^2 = (dF^1)^2 + (dF^2)^2 + (dF^3)^2$, which in spherical coordinates becomes $ds^2 = (df^1)^2 + \sin^2 f^1\,(df^2)^2$. The latter expression for ds is the *Riemannian metric* in S^2 that is induced from the Euclidean metric in $\Re^3$. It can also be expressed more generally as

$$ds^2 = \sum_{\alpha,\beta}^{2} h_{\alpha\beta}(\mathbf{f})\, df^i\, df^j \qquad (1)$$

where $h_{\alpha\beta}(\mathbf{f})$ is a symmetric positive-definite quadratic form. The sphere is parametrized by $0 \leq f^1 \leq 2\pi$, $0 \leq f^2 \leq \pi$, and its surface area is given by $\int_0^{\pi} \int_0^{2\pi} \sqrt{\det h_{\alpha\beta}(\mathbf{f})}\, df^1\, df^2$, which produces the desired result of 4π.

The above calculation is elementary, but it illustrates the dependence of the surface area (as well as lengths and angles of curves on S^2) on the choice of Riemannian metric. If incremental arc length in $\Re^3$ had been defined more generally as $ds^2 = \sum a_{\alpha\beta}(\mathbf{F})\, dF^i\, dF^j$, where $a_{\alpha\beta}$ is any smooth symmetric positive-definite quadratic form, one would obtain a different Riemannian metric $h_{\alpha\beta}$ on the sphere, leading to a different value for its surface area. In fact, one could even bypass $\Re^3$ altogether and instead choose a metric directly on the sphere. The question of choosing Riemannian metrics is particularly important in defining manipulability for closed chains; in this case the joint space manifold forms a surface, and the choice of metric on this surface reflects properties of the actuators driving the closed chain.

Returning to our spherical mechanism example, on S^2 the induced metric given by $(df^1)^2 + \sin^2 f^1 \; (df^2)^2$ is physically reasonable in that it preserves the isotropy of the workspace. In the joint space T^2, however, there is no physically compelling reason to choose the Euclidean metric $ds^2 = (du^1)^2 + (du^2)^2$. A more general "flat" metric is of the form $ds^2 = \epsilon_1(du^1)^2 + \epsilon_2(du^2)^2$, where $\epsilon_1\epsilon_2 = 1$ to fix the volume of T^2. This metric can be given the following interpretation. If all of the joint actuators are assumed to be identical, and we wish to place single gears between each actuator and the corresponding joint, then $\sqrt{\epsilon_i}$ can be interpreted as the gear ratio between joint i and its actuator. If we no longer assume identical joint actuators, then each $\sqrt{\epsilon_i}$ reflects the maximum velocity attainable by the joint i actuator. For actuator sizing problems the ϵ_i can be regarded as a design parameter to be optimized with respect to some performance measure.

In analyzing the manipulability of this mechanism, the ideal situation would be if the forward kinematic map f from T^2 to S^2 were an isometry everywhere. That is, the distance between any two points in T^2 would be equal to the distance between their image points in S^2, and the angle between any two intersecting curves in T^2 would be equal to the angle between their image curves in S^2. The mathematical condition for f to be an isometry is that $\mathbf{J}^{\mathrm{T}}\mathbf{H}\mathbf{J} = \mathbf{G}$, where $\mathbf{J}$ is the Jacobian of f, and $\mathbf{G}$ and $\mathbf{H}$ are matrix representations of g_{ij} (the metric on T^2) and $h_{\alpha\beta}$ (the metric on S^2), respectively. One of the important theorems of Gauss states that when two surfaces are of different Gaussian curvature as is the case here, then no global isometry exists. Since the ideal situation is when $\mathbf{J}^{\mathrm{T}}\mathbf{H}\mathbf{J} = \mathbf{G}$, or equivalently $\mathbf{J}^{\mathrm{T}}\mathbf{H}\mathbf{J}\mathbf{G}^{-1} = \mathbf{I}$, one measure of how closely f approximates an isometry is to see how "clustered" the eigenvalues of $\mathbf{J}^{\mathrm{T}}\mathbf{H}\mathbf{J}\mathbf{G}^{-1}$ are; clearly when all the eigenvalues are 1 the equality condition for an isometry will be satisfied.

We first fix some notation. Let $\mathcal{M}$ denote the manifold corresponding to the joint space of the mechanism, and $\mathbf{G}$ the Riemannian metric on $\mathcal{M}$. For an m-revolute open chain $\mathcal{M}$ will be an m-dimensional torus, and $\mathbf{G}$ a diagonal matrix (in terms of the standard coordinates for a flat torus) whose entries will be proportional to the maximum velocity of the corrresponding actuator. Let $\mathcal{N}$ denote the n-dimensional manifold corresponding to the end-effector space of the mechanism, and $\mathbf{H}$ the Riemannian metric on $\mathcal{N}$. For example, if orientations are ignored, then $\mathcal{N}$ will be $\Re^2$ for a planar mechanism, and $\Re^3$ for a spatial mechanism. The Riemannian metric in this situation will be chosen to be the Euclidean metric, *i.e.*, $\mathbf{H} = \mathbf{I}$.

Manipulability can now be defined in a coordinate-invariant way as follows. First, any symmetric function of the roots of $\det(\mathbf{G}\lambda - \mathbf{J}^{\mathrm{T}}\mathbf{H}\mathbf{J}) = 0$ (or, in coordinate-free language, the proper values of the pullback metric f^*h—see, e.g., Eells and Sampson (1964)) is a coordinate-invariant function defined on the Riemannian manifold $\mathcal{M}$. At a regular point the Jacobian of f by definition will have maximal rank, and the proper values of f^*h will have exactly n nonzero values (assuming $m \geq n$); we label these, in descending order, as $\lambda_1, \lambda_2, \ldots, \lambda_n$. Various symmetric functions of these proper values can now be constructed to reflect kinematic manipulability. For example, since singularities are marked by λ_n going to zero, maximizing λ_n is one possibility for a local kinematic manipulability measure that penalizes singular configurations. A condition number-based manipulability measure is

$$c(f) = \frac{\lambda_1}{\lambda_n} \tag{2}$$

while the generalization of Yoshikawa's manipulability measure $\sqrt{\det(\mathbf{J}\mathbf{J}^{\mathrm{T}})}$ is

$$v(f) = (\lambda_1 \cdots \lambda_n)^{\frac{1}{2}} \tag{3}$$

To obtain a global measure of manipulability one can integrate any of the various local manipulability measures over the joint space manifold $\mathcal{M}$:

$$E(f) = \int_{\mathcal{M}} \mu(f)\, dV \tag{4}$$

where $\mu(f)$ is an appropriate local measure, and $dV = \sqrt{\det \mathbf{G}}\, dx_1 \cdots dx_m$ is the volume form on M. One interesting choice of $\mu(f)$ is given by $\mu(f) = \mathrm{Tr}(\mathbf{J}^{\mathrm{T}}\mathbf{H}\mathbf{J}\mathbf{G}^{-1})$; this leads to the integral functional for harmonic maps, whose minima can be viewed as the mapping that best approximates a global isometry, see Eells and Sampson (1964), Park (1995a).

So far we have assumed that the dimension of $\mathcal{M}$ is equal to or less than the dimension of $\mathcal{N}$. The above results also apply to the redundant case, i.e., when $m \geq n$. Regardless of whether the mechanism is redundant or not, it should be intuitively clear from smoothness conditions that if we map, via the Jacobian, a unit sphere on the tangent space of $\mathcal{M}$, the result will be an ellipsoid on the tangent space of $\mathcal{N}$. To state this more precisely, let $\mathcal{S} = \{\mathbf{u} \in \Re^m | \, \mathbf{u}^{\mathrm{T}}\mathbf{G}\mathbf{u} = 1\}$, and $\mathcal{E} = \mathbf{J}\mathcal{S}$, i.e., $\mathcal{E}$ is the image of $\mathcal{S}$ under the linear map $\mathbf{J}$. Then relative to the Riemannian metric $\mathbf{H}$ on $\mathcal{N}$, the principal axes of $\mathcal{E}$ are determined by the eigenvalues and eigenvectors of $\mathbf{J}\mathbf{G}^{-1}\mathbf{J}^{\mathrm{T}}\mathbf{H}$.

3 Manipulability for Closed Chains

The concept of manipulability described above was formulated in the setting of general Riemannian manifolds, with a coordinate-invariant manipulability function defined on the configuration space manifold. Because of its generality, this formulation is equally valid for open and closed chains; the additional issue to consider in the latter case is the selection of the joint space Riemannian metric. We now discuss the issue of metrics, along with the general procedure for formulating closed chain manipulability.

Consider a mechanism with a total of k prismatic and revolute joints, possibly containing closed loops. As before denote the m-dimensional configuration space manifold by $\mathcal{M}$, where $m \leq k$; for our purposes $\mathcal{M}$ can be viewed as an m-dimensional surface in $\Re^k$. Let $\mathbf{x} = (x^1, \ldots, x^k)$ and $\mathbf{u} = (u^1, \ldots, u^m)$ denote local coordinates on $\Re^k$ and $\mathcal{M}$, respectively, so that a point $\mathbf{x} \in \mathcal{M}$ can be written as a function of $\mathbf{u}$, i.e., $\mathbf{x} = \mathbf{x}(\mathbf{u})$. Suppose a tool reference frame is attached to some point on the mechanism, and denote its configuration manifold by $\mathcal{N}$. Let $\mathbf{f} = (f^1, \ldots, f^n)$ denote local coordinates for $\mathcal{N}$, where $n \leq 6$. For general spatial mechanisms $\mathcal{N}$ will be some subset of the group of spatial displacements, or the Special Euclidean Group SE(3). For planar mechanisms $\mathcal{N}$ will be given by the planar displacements SE(2), while for spherical mechanisms $\mathcal{N}$ will be SO(3). For certain applications one may also wish to ignore orientations and consider position only; in this case $\mathcal{N}$ can be taken to be a subset of $\Re^2$ or $\Re^3$.

We now discuss the choice of an appropriate Riemannian metric for $\mathcal{M}$, followed by that for $\mathcal{N}$. Rather than choose a metric directly on $\mathcal{M}$, we first choose a metric for $\Re^k$, and project this metric onto $\mathcal{M}$. The metric on the ambient space $\Re^k$ can be given the same physical interpretation as for the open chain case. That is, the metric is taken to be of the form $ds^2 = \epsilon_1(dx^1)^2 + \ldots + \epsilon_k(x^k)^2$, where each ϵ_i can be interpreted as the gear ratio between the ith actuator and ith joint. In the event that the joint is not actuated, ϵ_i is then set to zero. This choice of metric can be justified from virtual work considerations. Recalling that work is force times distance, if a particular joint is not externally actuated then its contribution to the total work should be zero regardless of any displacement that the joint may have undergone. Setting ϵ_i to zero achieves the desired result by ignoring the joint displacement. This is also consistent with the fact that for lossless mechanical systems, constraint forces do no work.

Denote the metric on $\Re^k$ by $\mathbf{E} = \mathrm{Diag}[\epsilon_1 \ldots \epsilon_k]$. Note that because some of the ϵ_i may be zero, the metric may only be positive semi-definite (and hence strictly speaking only a pseudo-Riemannian metric). The metric on $\mathcal{M}$ is then obtained by projecting $\mathbf{E}$ onto $\mathcal{M}$: since $d\mathbf{x} = \nabla_{\mathrm{u}}\mathbf{x} \cdot d\mathbf{u}$, where

$$\nabla_{\mathrm{u}}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{u}} \tag{5}$$

it follows that

$$\begin{aligned} ds^2 &= \frac{1}{2} d\mathbf{x}^{\mathrm{T}} \mathbf{E}\, d\mathbf{x} && (6) \\ &= \frac{1}{2} d\mathbf{u}^{\mathrm{T}} (\nabla_{\mathrm{u}}\mathbf{x})^{\mathrm{T}} \mathbf{E} (\nabla_{\mathrm{u}}\mathbf{x}) d\mathbf{u} && (7) \end{aligned}$$

The projected metric on $\mathcal{M}$, denoted $\mathbf{G}$, is therefore $\mathbf{G} = (\nabla_{\mathrm{u}}\mathbf{x})^{\mathrm{T}}\mathbf{E}(\nabla_{\mathrm{u}}\mathbf{x})$.

The metric on $\mathcal{N}$, which we denote $\mathbf{H}$, is also chosen from physical considerations. In the event that $\mathcal{N}$ is some proper subset of SO(3) or Euclidean space, there exists a natural choice of Riemannian metric, given by the standard Euclidean inner product for velocities and angular velocities (see Loncaric 1985, Park 1995b). When $\mathcal{N}$ is taken to be either SE(2) or SE(3), however, there now exists a one-parameter family of suitable Riemannian metrics parametrized by the choice of length scale for physical space. Since there is no natural length scale for physical space, there is some arbitrariness involved in the choice of metric on SE(3); Park (1995b) suggests some ways in which to choose a metric that takes into account the given task. In general, however, this arbitrariness is an unavoidable consequence of the geometry of SE(3).

Denote the forward kinematics $\mathbf{f} : \mathcal{M} \to \mathcal{N}$ in local coordinates as $\mathbf{f}(\mathbf{u})$, and its Jacobian by $\mathbf{J} = \nabla_{\mathrm{u}}\mathbf{f}$. The manipulability ellipsoid is then determined by the eigenvalues and eigenvectors of $\mathbf{J}\mathbf{G}^{-1}\mathbf{J}^{\mathrm{T}}\mathbf{H}$. In the following section we investigate the manipulability of various linkages using our geometric framework developed above, but before doing so we outline the computational procedure for determining manipulability ellipsoids that does not require explicit knowledge of $\mathbf{x}(\mathbf{u})$ and its derivatives.

First, recall that k is the total number of joints in the mechanism, while m is the mechanism's kinematic degrees of freedom. In what follows we restrict our attention to spatial mechanisms, in which case $\mathcal{N} = \mathrm{SE}(3)$; the procedure can be suitably modified for planar or spherical mechanisms.

- **Algebraic Constraint Equations**: Let the kinematic loop constraints be written

$$\mathbf{h}_i(\mathbf{x}) = \mathbf{I}, \quad i = 1, \ldots, p \tag{8}$$

  where each $\mathbf{h}_i : \Re^k \to \mathrm{SE}(3)$ represents a closed loop of the mechanism. Note that each equation represents six constraints on $\mathbf{x}$; assuming each $\mathbf{h}_i(\mathbf{x})$ represents an independent kinematic constraint, p must satisfy $6p = k - m$.

- **Jacobian J**: Without loss of generality, let $\mathbf{u} = (x_1, \ldots, x_m)$ be the set of local coordinates for the mechanism (the joints can be renumbered accordingly to satisfy this condition). Assuming an end-effector frame has been attached, find the Jacobian $\mathbf{J}(\mathbf{x})$ that satisfies

$$\begin{bmatrix} \omega \\ v \end{bmatrix} = \mathbf{J}(\mathbf{x}) \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} \tag{9}$$

  where (ω, v) denotes the generalized velocity of the end-effector frame, expressed with respect to either the inertial or end-effector frame (Note: the choice of reference frame determines whether the Riemannian metric on SE(3) is right-invariant or left-invariant—see below).

- **Task Space Manifold Metric H**: Choose the metric $\mathbf{H}$; one possible choice is $\mathbf{H} = \mathbf{I}$, which recall is parametrized by choice of length scale for physical space. If $\mathbf{J}$ is derived with respect to the inertial frame, then $\mathbf{H}$ represents a right-invariant metric, while if $\mathbf{J}$ is derived with respect to the end-effector frame, then $\mathbf{H}$ represents a left-invariant metric.

- **Joint Space Manifold Metric G**: Recall that the metric $\mathbf{G} = (\nabla_u \mathbf{x})^T \mathbf{E} (\nabla_u \mathbf{x})$, where $\mathbf{E} = \text{Diag}[\epsilon_1 \cdots \epsilon_k]$. If the ith joint is actuated, set ϵ_i to a positive constant that is proportional to the size of the actuator; otherwise, set $\epsilon_i = 0$. To find $\nabla_u \mathbf{x})$, first suppose that $\mathbf{h}_i : \Re^k \to SE(3)$ now represents the forward kinematics of an open chain with its end-effector fixed and stationary, and derive the associated Jacobian:

$$
\begin{aligned}
0 &= (\nabla_x \mathbf{h})\dot{\mathbf{x}} && (10) \\
&= \mathbf{A}_i(\mathbf{x}) \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} + \mathbf{P}_i(\mathbf{x}) \begin{bmatrix} \dot{x}_{m+1} \\ \vdots \\ \dot{x}_k \end{bmatrix} && (11)
\end{aligned}
$$

for $i = 1, \ldots, p$. If we now stack this set of p equations into a single equation, i.e.,

$$
\begin{aligned}
0 &= \begin{bmatrix} \mathbf{A}_1(x) \\ \vdots \\ \mathbf{A}_p(x) \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} + \begin{bmatrix} \mathbf{P}_1(x) \\ \vdots \\ \mathbf{P}_p(x) \end{bmatrix} \begin{bmatrix} \dot{x}_{m+1} \\ \vdots \\ \dot{x}_k \end{bmatrix} && (12) \\
&= \mathcal{A}(\mathbf{x}) \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} + \mathcal{P}(\mathbf{x}) \begin{bmatrix} \dot{x}_{m+1} \\ \vdots \\ \dot{x}_k \end{bmatrix} && (13)
\end{aligned}
$$

then $\mathcal{A}(\mathbf{x}) \in \Re^{(k-m)\times m}$ while $\mathcal{P}(\mathbf{x}) \in \Re^{(k-m)\times(k-m)}$. Provided the mechanism is not at a kinematic singularity, $\mathcal{P}(\mathbf{x})$ will be invertible. Rearranging,

$$
\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{m\times m} \\ -\mathcal{P}^{-1}(\mathbf{x})\mathcal{A}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} \qquad (14)
$$

where $\mathbf{I}_{m\times m}$ denotes the $m \times m$ identity matrix. It follows that

$$
\mathbf{G} = \begin{bmatrix} \mathbf{I}_{m\times m} \\ -\mathcal{P}^{-1}(\mathbf{x})\mathcal{A}(\mathbf{x}) \end{bmatrix}^T \mathbf{E} \begin{bmatrix} \mathbf{I}_{m\times m} \\ -\mathcal{P}^{-1}(\mathbf{x})\mathcal{A}(\mathbf{x}) \end{bmatrix} \qquad (15)
$$

- **Manipulability Ellipsoid**: The principal axes of the ellipsoid are now determined from the eigenvalues and eigenvectors of $\mathbf{J}\mathbf{G}^{-1}\mathbf{J}^T\mathbf{H}$, whose components can be constructed following the computational procedure outlined above.

4 Manipulability Examples

In this section we examine the manipulability of several closed chains: a planar five-bar linkage, a planar six-bar linkage, and a spherical six-bar linkage. For the planar cases we ignore orientation and only consider the Cartesian position of the tip.

Planar Five-Bar Linkage

The five-bar linkage is shown in Figure 2. We assume all the link lengths to be equal, with the two fixed pivots also separated by a link length. Two cases are considered. In the first case we assume that the two base joints are actuated (observe that this mechanism has two degrees of freedom), while in the second case the two neighboring middle joints are actuated. In the figures the actuated joints are shaded, while the passive joints are indicated by unfilled circles. We draw the manipulablility ellipsoid at 3 points in the workspace for both cases. From a visual inspection it is clear that for the given set of workspace points, placing the actuators at the middle joints provides better kinematic manipulability.

Planar Six-Bar Linkage

To illustrate how our formalism can uniformly handle both overactuated and exactly actuated mechanisms, consider the three degree-of-freedom six-bar linkage of Figure 3. Because we ignore orientation and consider only planar position, this mechanism is kinematically redundant. Two cases are considered. In the first case the mechanism is overactuated, by placing actuators at the four joints closest to the fixed pivots (shown as filled circles). In the second case only three of the mechanism's joints are actuated. Once again we draw manipulability ellipsoids for three points in the workspace. As expected, for the overactuated case the manipulability ellipsoid is aligned along the x- and y-axes when the mechanism is in a symmetric posture. However, when only three of the joints are actuated the ellipsoid's orientation is skewed for the same posture.

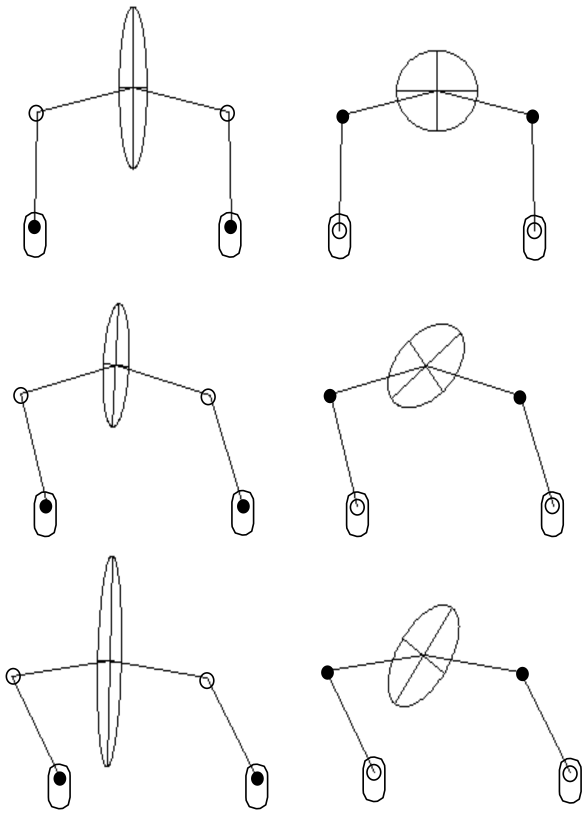

Figure 2: Manipulability ellipsoids for the five-bar linkage. Filled circles represent actuated joints.

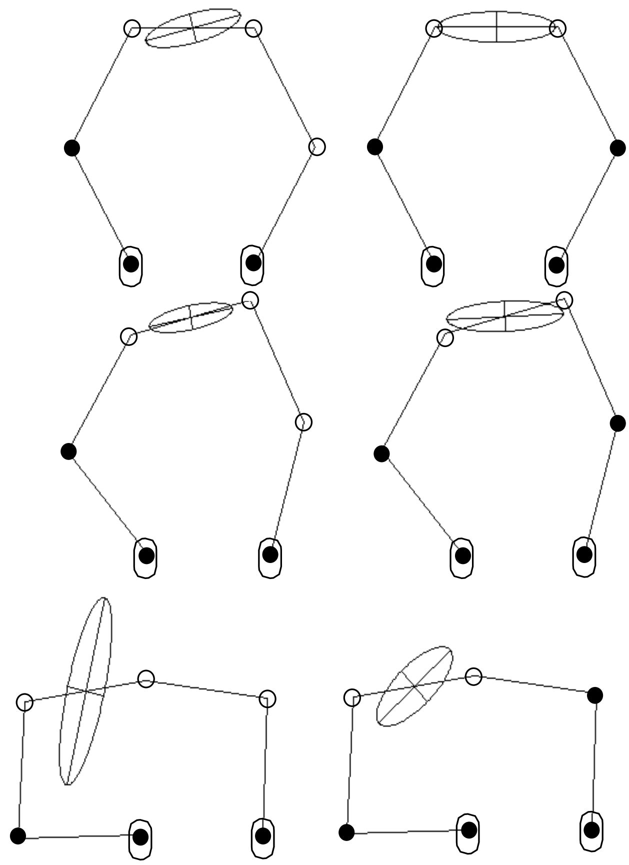

Figure 3: Manipulability ellipsoids for the six-bar linkage. Filled circles represent actuated joints.

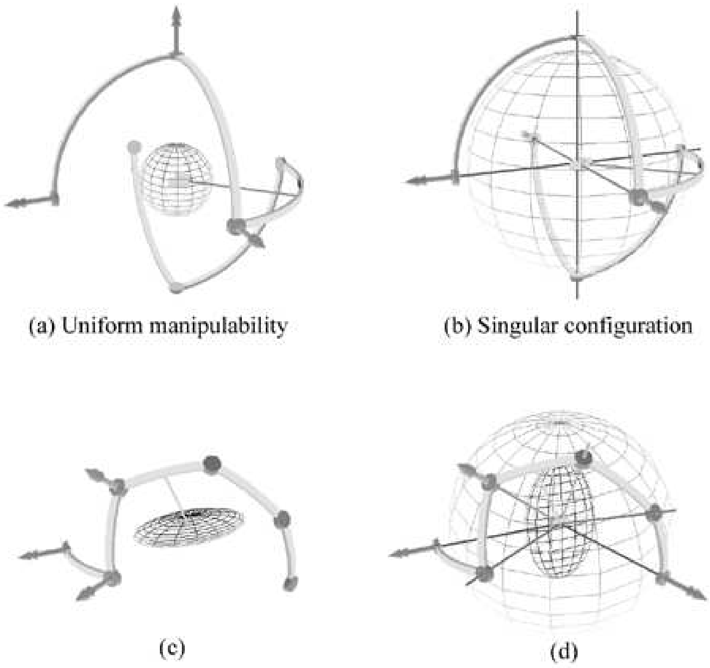

Figure 4: Manipulability ellipsoids for the spherical six-bar linkage. The arrows indicate actuated joints.

Spherical Six-Bar Linkage

Figures 4a and 4b depict a three degree-of-freedom spherical six-bar linkage, whose kinematic constraint equations are given by

$$e^{\mathbf{A}_1\theta_1} e^{\mathbf{A}_2\theta_2} \cdots e^{\mathbf{A}_6\theta_6} = \mathbf{I} \tag{16}$$

where the $\mathbf{A}_i$ are the following 3×3 skew-symmetric matrices:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{A}_5 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_6 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Each matrix exponential is a rotation matrix given by

$$e^{\mathbf{A}_i \theta_i} = \mathbf{I} + \sin\theta \mathbf{A}_i + (1 - \cos\theta)\mathbf{A}_i^2 \tag{17}$$

In both cases we attach actuators to exactly three of the six revolute joints; the actuated joints are indicated by the arrows. The end-effector is rigidly attached to the middle link, and coincides with the inertial frame when the mechanism is in its home configuration. In Figure 4a we see that the manipulability ellipsoid is perfectly uniform, while in Figure 4b the ellipsoid collapses, indicating a kinematic singularity. Geometrically, kinematic singularities of the spherical six-bar linkage occur when the axes of rotation of the actuated joints do not span $\Re^3$; this can be straightforwardly verified by a direct calculation.

Figures 4c and 4d depict a different six-bar spherical linkage, this time with the $\mathbf{A}_i$ in the kinematic constraint equations given by

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & -0.1736 & 0.6330 \\ 0.1736 & 0 & -0.7544 \\ -0.6330 & 0.7544 & 0 \end{bmatrix}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & -0.7071 & 0.3536 \\ 0.7071 & 0 & -0.6124 \\ -0.3536 & 0.6124 & 0 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 0 & -0.866 & 0.5 \\ 0.866 & 0 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_5 = \begin{bmatrix} 0 & -0.5 & 0.75 \\ 0.5 & 0 & 0.433 \\ -0.75 & -0.433 & 0 \end{bmatrix}, \quad \mathbf{A}_6 = \begin{bmatrix} 0 & 0 & 0.866 \\ 0 & 0 & 0.5 \\ -0.866 & -0.5 & 0 \end{bmatrix}$$

The actuated joints are again indicated by the arrows. Observe that the shape of the ellipsoid is strongly influenced by the choice of actuated joints. The three numbers shown below the ellipsoid indicate the lengths of the ellipsoid's principal axes. See, e.g., McCarthy (1997), Gosselin (1996) for examples of some practical spherical mechanisms.

5 Dynamic Manipulability

One of the attractive features of the proposed manipulability framework is that dynamic and inertial characteristics can be included in a natural way. Specifically, the metric $\mathbf{G}$ is defined to be the generalized mass matrix of the mechanism, and $\mathbf{H}$ to be the Riemannian metric corresponding to the kinetic energy of the manipulated object. Recall that the kinetic energy of a rigid body is defined by its mass m and inertia matrix $\mathbf{I}$: assuming the center of mass is moving with velocity $\mathbf{v}$, and the body is rotating with angular velocity $\boldsymbol{\omega}$, all expressed relative to the body-fixed frame, then its kinetic energy is given by $\langle(\boldsymbol{\omega},\mathbf{v}),(\boldsymbol{\omega},\mathbf{v})\rangle = \frac{1}{2}(m\mathbf{v}^{\mathrm{T}}\mathbf{v}+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{I}\boldsymbol{\omega})$. The kinetic energy $\langle,\cdot,\cdot\rangle$ therefore defines a (left-invariant) Riemannian metric on SE(3). Recall also that the dynamics of any holonomic mechanical system can be written in the form

$$\tau = \mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{N}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) \tag{18}$$

where $\boldsymbol{\theta}$ denotes a minimal set of generalized coordinates for the system. Choosing $\mathbf{M}(\boldsymbol{\theta})$ to be $\mathbf{G}$, and $\mathbf{H}$ to be the Riemannian metric corresponding to the kinetic energy of the manipulated object, leads to a coordinate-invariant formulation of dynamic manipulability.

To illustrate the dynamic manipulability concept, we determine the manipulability ellipsoid for two cooperating 6R PUMA 560 robots holding a common object. Such a system consists of 12 joints, and has 6 kinematic degrees of freedom. We label the manipulators a and b, and denote their corresponding joint vectors by $\boldsymbol{\theta}_a \in \Re^6$ and $\boldsymbol{\theta}_b \in \Re^6$. Choose the metric $\mathbf{E}$ for the ambient 12-dimensional space to be

$$\mathbf{E} = \begin{bmatrix} \mathbf{M}_a & 0 \\ 0 & \mathbf{M}_b \end{bmatrix} \tag{19}$$

where $\mathbf{M}_a$ and $\mathbf{M}_b$ are respectively the generalized mass matrices of manipulators a and b. The values for the generalized mass matrix of the PUMA 560 can be found in Armstrong et al (1986). Choose $\boldsymbol{\theta}_a$ to be the set of generalized coordinates for the system; the metric $\mathbf{G}$ is then the generalized mass matrix for the system expressed in terms of the chosen generalized coordinates. Let $\mathbf{J}_a(\boldsymbol{\theta}_a)$ and $\mathbf{J}_b(\boldsymbol{\theta}_b$ be the Jacobians of the respective manipulators, so that $\mathbf{J}_a = \mathbf{J}_b$. Following the procedure outlined in the previous section,

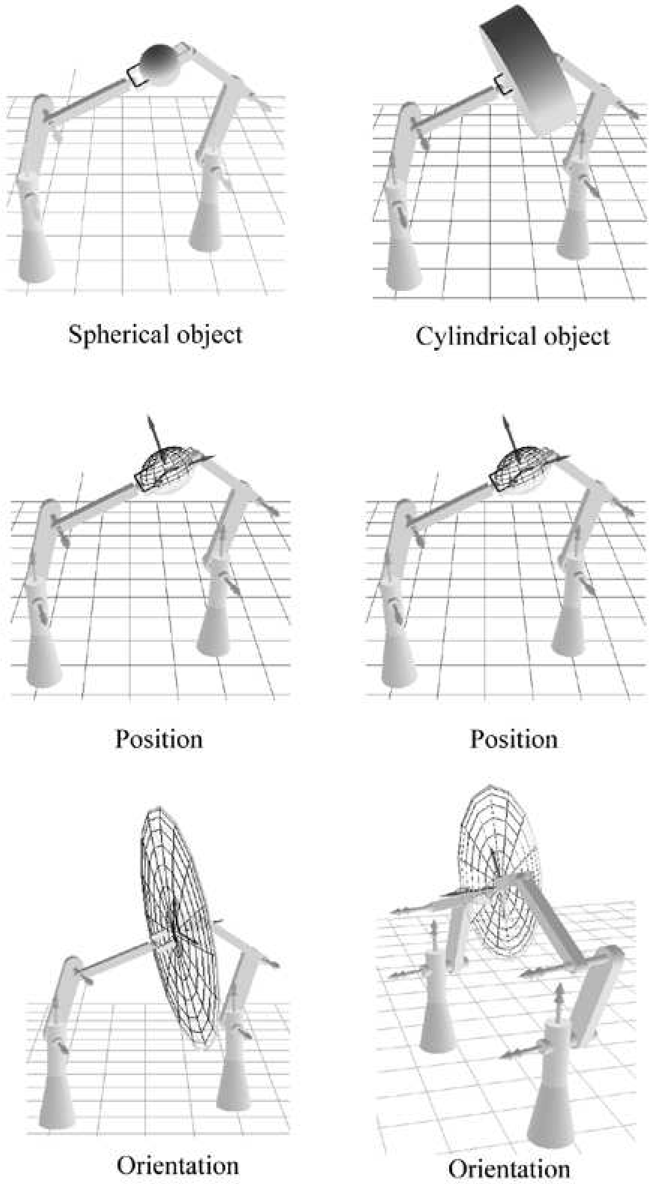

Figure 5: Dynamic manipulability ellipsoids for two cooperating PUMA 560 manipulators.

$\mathbf{G}$ can be derived as

$$\mathbf{G} = \mathbf{M}_a + (\mathbf{J}_b^{-1}\mathbf{J}_a)^{\mathrm{T}}\mathbf{M}_b(\mathbf{J}_b^{-1}\mathbf{J}_a) \qquad (20)$$

We consider two different objects: a ball of mass $m = 1$kg and radius $r = 0.1$m, and a cylinder of mass m = 1 kg, radius $r = 0.3$m, and height $h = 0.2$m. For the ball the moments of inertia are $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$, whie for the cylinder (assume the z-axis of the body frame is directed along the height of the cylinder) the moments of inertia are given by $I_{xx} = \frac{1}{2}mr^2$, $I_{yy} = I_{zz} = \frac{1}{12}m(3r^2 + h^2)$. The products of inertia are zero for both objects. The manipulability ellipsoids are then determined from the quadratic form

$$\mathbf{J}\mathbf{G}^{-1}\mathbf{J}^{\mathrm{T}}\mathbf{H} = \mathbf{J}_a(\mathbf{M}_a + (\mathbf{J}_b^{-1}\mathbf{J}_a)^{\mathrm{T}}\mathbf{M}_b(\mathbf{J}_b^{-1}\mathbf{J}_a))^{-1}\mathbf{J}_a^{\mathrm{T}}\mathbf{H} \qquad (21)$$

As indicated previously, because of the lack of a natural metric on SE(3), the six-dimensional ellipsoids generated from the above quadratic form will depend on the choice of length scale for physical space (this essentially can be traced to the fact that mixing units for linear and angular velocities results in a physically inconsistent quantity). Rather, it is more useful to consider separately the orientation and position manipulability of the system. Figure 5 illustrates the position and orientation manipulablity ellipsoids for the cooperating system at an arbitrary fixed posture for the two different objects. Observe that if we choose $\mathbf{M}_a = \mathbf{M}_b = \mathbf{H} = \mathbf{I}$, then this is equivalent to the kinematic manipulability case with the length-scale dependent Riemannian metric on SE(3).

6 Conclusions

In this paper we have presented a geometric analysis of manipulability for general mechanisms. The methods of Riemannian geometry provide a unified framework in which to investigate manipulability for the full range of possible mechanisms: redundant, nonredundant, exactly actuated, overactuated, serial, parallel, and hybrid. For example, the actuator characteristics are reflected in the choice of a Riemannian metric for the joint and tool frame configuration space manifolds, or one can even include inertial parameters in the Riemannian metric to obtain a formulation for dynamic manipulablility. Most importantly, the manipulability definitions are independent of the choice of parametrization for these two spaces, as well as the kinematic mapping. In a companion paper we employ the same mathematical framework to classify and analyze kinematic singularities for closed chains.

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