Physically Based Animation 2008

Hair Simulation 2

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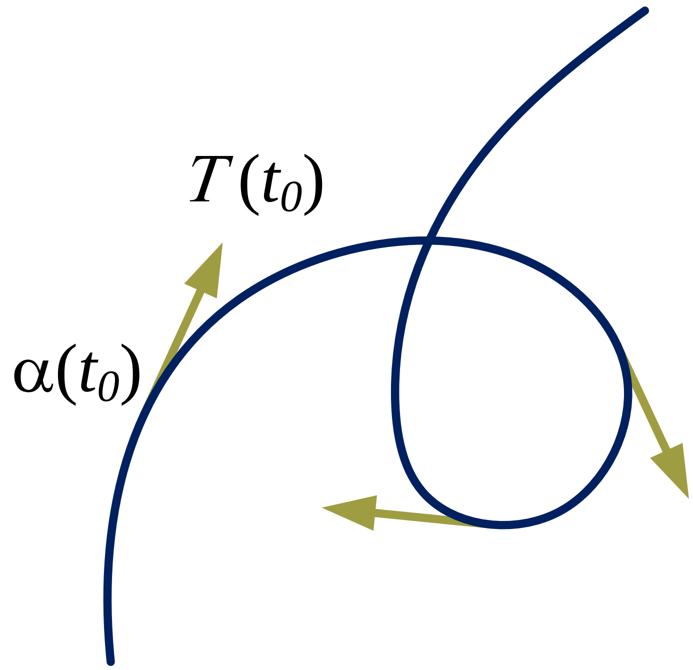
Curve Representation

- Regular curve in $\mathbf{R}^3$
  - A function α :(a,b)→$\mathbf{R}^3$
  - $\frac{d\alpha}{dt} \neq 0 \quad \forall t \in (a,b)$
- Tangent vector field

$$T(t) = \frac{d\alpha / dt}{\| d\alpha / dt \|}$$

  - Geometric quantity
    - Depends only on the shape and not the parameterization

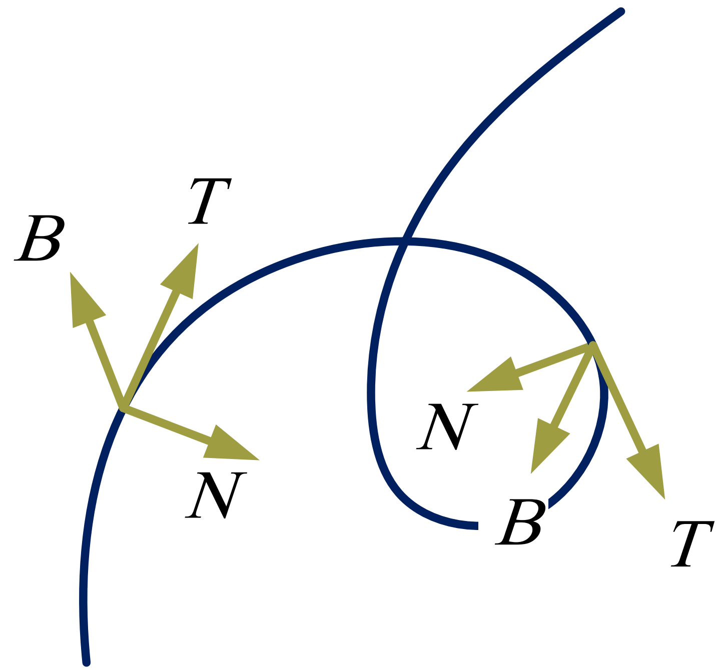


Unit Speed Curve

- Arc length
  - $s = s(t) = \int_0^t \| d\alpha / dt \| \, dt$
  - $T(t) = \dfrac{d\alpha}{ds}$
  - We are interested only in a shape of the curve
    - We don’t care about parameterization
  - Arc length can be a ***nice*** parameter

Frenet-Serret Frame

- ***T*** : Tangent vector
  - Along velocity
- ***N*** : Normal vector
  - On a plane of tangent vectors
  - $N(s) = T'(s) / \| T'(s) \|$
- ***B*** : Bi-normal vector
  - $B(s) = T(s) \times N(s)$
- { ***T***, ***N***, ***B*** } define a coordina te frame along the curve



Curvature and Torsion

- Curvature
  - $\kappa(s) = \| T'(s) \|$
  - Measure of bending
    - Curvature of a straight line = 0
    - Curvature of a circle = const.
- Torsion
  - $\tau(s) = -B'(s) \cdot N(s) = \dfrac{T \cdot (T' \times T'')}{\kappa^2}$
  - Measure of twisting
    - Torsion of a curve on a plane = 0
    - Torsion of a helix = const.

Potential Energy for a Curve

- Account bending and twisting

$$E = \frac{1}{2}\int \left(k_{\kappa}(\kappa - \kappa_{r})^{2} + k_{\tau}(\tau - \tau_{r})^{2}\right) ds$$

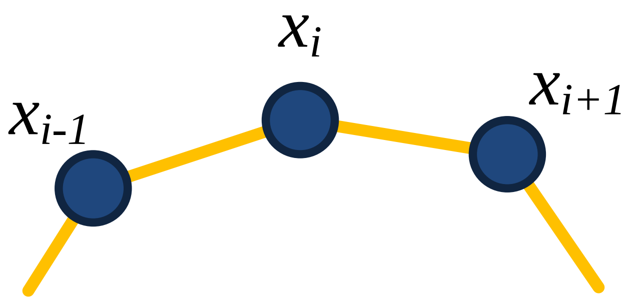
- Note that the potential energy is a function of o nly a shape of the curve

Numerical Differentiation

- Given a set of function values at discrete points, how to find derivatives of the function?
- From Taylor approximation,
  - Forward difference $f'(x) = \left(f(x+h) - f(x)\right)/h$
  - Backward difference $f'(x) = \left(f(x) - f(x-h)\right)/h$
  - Truncation error = $O(hf'')$
- Combining two
  - Central difference $$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$
  - Truncation error = $O(h^2 f''')$
  - *Symmetry* helps

Discretization of a Hair Strand

- Numerical Differentiation
  - $$T_i = \frac{x_{i+1} - x_{i-1}}{\| x_{i+1} - x_{i-1} \|}$$
  - $$T_i' = \frac{T_{i+1/2} - T_{i-1/2}}{\| x_{i+1/2} - x_{i-1/2} \|}$$
    $$= \frac{2}{\| x_{i+1} - x_{i-1} \|} \left(\frac{x_{i+1} - x_i}{\| x_{i+1} - x_i \|} - \frac{x_i - x_{i-1}}{\| x_i - x_{i-1} \|} \right)$$



Discretization of a Hair Strand

- $$T_i'' = \frac{1}{2}\left(T_{i-1/2}'' + T_{i+1/2}''\right)$$

  $$= 2\left(\frac{T_i' - T_{i-1}'}{\| x_{i+1} + x_i - x_{i-1} - x_{i-2} \|} + \frac{T_{i+1}' - T_i'}{\| x_{i+2} + x_{i+1} - x_i - x_{i-1} \|}\right)$$

- $$\kappa_i = \| T_i' \| \qquad \tau_i = \frac{T_i \cdot (T_i' \times T_i'')}{\kappa_i^{\,2}}$$

- Discretized potential energy

$$E = \frac{1}{2}\sum_i \left(k_\kappa(\kappa_i - \bar{\kappa}_i)^2 + k_\tau(\tau_i - \bar{\tau}_i)^2\right)\left(\frac{\| x_i - x_{i-1} \| + \| x_{i+1} - x_i \|}{2}\right)$$

Simplification

- Assume a fixed length of $\Delta l = \| x_i - x_{i-1} \| = \| x_{i+1} - x_i \| = \cdots$
  - Then

$$T_i = \frac{e_i}{\| e_i \|}$$

$$T_i' = \frac{2c_i}{\Delta l \| e_i \|}$$

$$T_i'' = \frac{1}{\Delta l^2} \left(\frac{c_{i+1}}{\| e_{i+1} \|} - \frac{c_{i-1}}{\| e_{i-1} \|} \right)$$

, where $e_i = x_{i+1} - x_{i-1}$ $\quad c_i = x_{i+1} - 2x_i + x_{i-1}$

Bending and Twisting Force

- $$\kappa_i = \frac{2 \| c_i \|}{\Delta l \| e_i \|}$$

  $$\frac{\partial \kappa_i}{\partial x_i} = -\frac{2\kappa_i}{\| c_i \|^2} c_i$$

  $$\frac{\partial \kappa_i}{\partial x_{i+1}} = \frac{\kappa_i}{\| c_i \|^2} c_i - \frac{\kappa_i}{\| e_i \|^2} e_i$$

  $$\frac{\partial \kappa_i}{\partial x_{i+1}} = \frac{\kappa_i}{\| c_i \|^2} c_i + \frac{\kappa_i}{\| e_i \|^2} e_i$$

Bending and Twisting Force

- $\alpha_i = \dfrac{1}{2\Delta l \, \| c_i \|^2} \quad d_i = e_i \times c_i \quad \tau_i = \alpha_i \, e_i \cdot \left(c_i \times \left(\dfrac{c_{i+1}}{\| e_{i+1} \|} - \dfrac{c_{i-1}}{\| e_{i-1} \|} \right) \right)$

$$\frac{\partial \tau_i}{\partial x_i} = -\frac{\partial \tau_i}{\partial x_{i-2}} - \frac{\partial \tau_i}{\partial x_{i-1}} - \frac{\partial \tau_i}{\partial x_{i+1}} - \frac{\partial \tau_i}{\partial x_{i+2}}$$

$$\frac{\partial \tau_i}{\partial x_{i+1}} = -\frac{2\tau_i}{\| c_i \|^2} c_i - \alpha_i \left((e_i - c_i) \times \left(\frac{c_{i+1}}{\| e_{i+1} \|} - \frac{c_{i-1}}{\| e_{i-1} \|} \right) + \frac{2}{\| e_{i+1} \|} d_i \right)$$

$$\frac{\partial \tau_i}{\partial x_{i-1}} = -\frac{2\tau_i}{\| c_i \|^2} c_i - \alpha_i \left((e_i + c_i) \times \left(\frac{c_{i+1}}{\| e_{i+1} \|} - \frac{c_{i-1}}{\| e_{i-1} \|} \right) - \frac{2}{\| e_{i-1} \|} d_i \right)$$

$$\frac{\partial \tau_i}{\partial x_{i+2}} = -\alpha_i \left(\frac{c_{i+1} \times d_i}{\| e_{i+1} \|^3} e_{i+1} - \frac{1}{\| e_{i+1} \|} d_i \right)$$

$$\frac{\partial \tau_i}{\partial x_{i-2}} = -\alpha_i \left(\frac{c_{i-1} \times d_i}{\| e_{i-1} \|^3} e_{i-1} + \frac{1}{\| e_{i-1} \|} d_i \right)$$

Bending and Twisting Force

- Potential energy due to bending and twisting

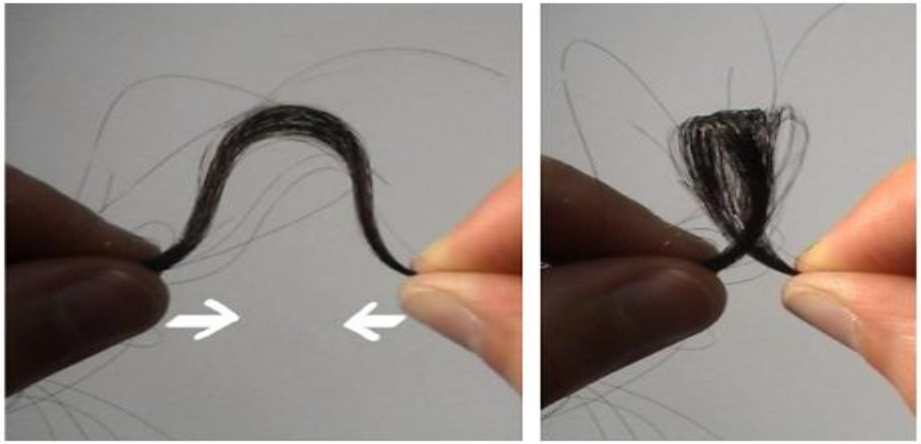
$$E = \frac{1}{2}\sum_i \left(k_\kappa(\kappa_i - \bar{\kappa}_i)^2 + k_\tau(\tau_i - \bar{\tau}_i)^2\right)\Delta l$$

- Bending and twisting force

$$f_i^T = -\frac{\partial E}{\partial x_i} = \sum_j \left(k_\kappa(\kappa_j - \bar{\kappa}_j)\frac{\partial \kappa_j}{\partial x_i} + k_\tau(\tau_j - \bar{\tau}_j)\frac{\partial \tau_j}{\partial x_i} \right)\Delta l$$

Bending and Twisting Instability

- Sudden collapse of the structure
  - If we apply compression, it bends
  - Beyond a certain threshold, it suddenly starts to coil at a small perturbation
  - Bending energy is transferred to twisting energy
  - Called the ***Buckling***



Relaxation

- Even with stiff spring, your hair will stretch
- Rectify the length every time step
  - Successive relaxation until convergence from bottom to top
- You can use less stiff spring
  - Means that you can use explicit integrator

Guide Hair

- Assume that you simulate 1M strands for rich hai r animation
- Plant 1K strands called ***guide hair***
- Simulate only guide hair
  - Resolve collision between hair/hair and hair/head
- The rest are just interpolated using the guide ha ir