Physically Based Animation 2008

Finite Element Method 1

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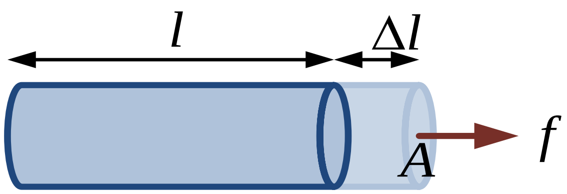
Mass Spring System

- Hair
  - 1-dim. chain of mass-spring system
- Cloth
  - 2-dim. Network of mass-spring system
- General 3-dim. Structure
  - We can do the similar things but
  - Difficult to deliver proper volumetric effects
  - Depends on the structure of the spring connections
  - Extremely difficult to find right coefficients

Hooke's law in 1-dim Case



$$\frac{f}{A} = E\frac{\Delta l}{l}$$

- E: Young's modulus, material property
- $\Delta l / l$: ***strain***, relative elongation
- f / A: ***stress***, force per area

Continuum Mechanics

- Generalize the Hooke's law to 3-dim. Case
  - x: ***material coordinates***, original position
  - $p(x)$: ***spatial coordinates***, deformed position
  - $u(x)$: ***deformation***, $u(x)=p(x)-x$
- Strain
  - Relative deformation of the material
  - Spatial derivatives of the deformation $u(x)$

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

Strain

- Consider the deformation $p(x)$ around $p(0)$

$$p(x) = p(0) + \nabla p \cdot x + O(\| x \|^2)$$

or

$$p(x) \approx p(0) + \begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \end{bmatrix} x$$

- Note that ∇p stretches, skews and rotates the material

Strain

- Measure of Deformation
  - Green's strain tensor

$$\varepsilon_G = \frac{1}{2}\left(\nabla p^T \nabla p - I\right)$$

$$= \frac{1}{2}\left((\nabla u + I)^T (\nabla u + I) - I\right)$$

$$= \frac{1}{2}\left(\nabla u + \nabla u^T + \nabla u^T \nabla u\right)$$

  - Cauchy's strain tensor

$$\varepsilon_C = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

Stress

- The force per unit area

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

- Force per area is

$$\frac{df}{dA} = \sigma \cdot n$$

, where n is the directional vector of measurement

Constitutive Law

- Relation between strain and stress

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

- E: Young's modulus
  - Elastic stiffness of the material
- V: Poisson's ratio
  - Resistance against volume change of the material

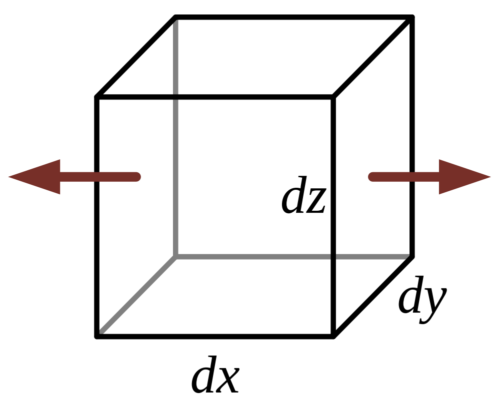
Equation of Motion

- Apply Newton's second law to the infinitesimal volumetric element

$$\rho \ddot{p} = f(x)$$

  - ρ : density
  - f : body force acting on the volume

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{bmatrix}_x dydz$$



$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{bmatrix}_{x+dx} dydz$$

Equation of Motion

- Body force in x axis

$$f_x = \left(\begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{bmatrix}_{x+dx,y,z} - \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{bmatrix}_{x,y,z} \right) dydz / dV = \begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial x} \\ \frac{\partial \sigma_{xy}}{\partial x} \\ \frac{\partial \sigma_{xz}}{\partial x} \end{bmatrix}$$

- In total,

$$f = f_x + f_y + f_z = \nabla \cdot \sigma$$

  - $\nabla \cdot$ represents the divergence, that is,

$$\nabla \cdot f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (f_x, f_y, f_z) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Finite Element Method

- Hyperbolic partial differential equation

$$\rho \ddot{p} = \nabla \cdot \sigma + f_{ext}$$

, where f_{ext} are externally applied body force such as g ravity and collision forces

- Discretize the deformable object into a finite set of elements of finite size
- We will use Cauchy’s strain tensor for linear P .D.E
  - At the cost of noticeable visual artifacts when larg e deformation

Tetrahedron as a Finite Element

- x_0, x_1, x_2, x_3 be the corners of the tetrahedron in the undeformed state
- p_0, p_1, p_2, p_3 be the same corners in the defor med state
- Express point inside the undeformed tetrahedr on

$$x = x_0 + b_1(x_1 - x_0) + b_2(x_2 - x_0) + b_3(x_3 - x_0)$$

$$= x_0 + \begin{bmatrix} | & | & | \\ x_1 - x_0 & x_2 - x_0 & x_3 - x_0 \\ | & | & | \end{bmatrix} b$$

Tetrahedron as a Finite Element

- Similarly

$$p(x) = p_0 + \begin{bmatrix} | & | & | \\ p_1 - p_0 & p_2 - p_0 & p_3 - p_0 \\ | & | & | \end{bmatrix} b$$

$$= p_0 + \begin{bmatrix} p_{10} & p_{20} & p_{30} \end{bmatrix} \begin{bmatrix} x_{10} & x_{20} & x_{30} \end{bmatrix}^{-1} (x - x_0)$$

, where $p_{ij} = p_i - p_j$

- Then

$$\nabla u = \nabla p - I = P - I$$

  - , where $P = \begin{bmatrix} p_{10} & p_{20} & p_{30} \end{bmatrix} \begin{bmatrix} x_{10} & x_{20} & x_{30} \end{bmatrix}^{-1}$

Tetrahedron as a Finite Element

- Hence the strain is

$$\varepsilon = \frac{1}{2}\left(\nabla u + \nabla u^T\right) = \frac{1}{2}\left(P + P^T\right) - I$$

and the stress is

$$\sigma = E\varepsilon$$

  - Note that E is 6X6 matrix
- Then the deformation force for face(0,1,2) is

$$f_{012} = \sigma n_{012} A_{012} = \sigma(p_{10} \times p_{20})$$

  - $f_0 = f_1 = f_2 = 1/3\, f_{012}$
  - Repeat it for face(0,1,3), face(0,2,3) and face(1,2,3)

Recipe

- Initialize X_i
- Simulation loop
  - For all tetrahedra
    - Compute P
    - Compute strain
    - Compute stress
    - For all faces
      - Compute forces
  - For all vertices
    - Integrate dynamics

Tetrahedral Division of a Cube

Elements	Nodes			
	1	2	3	4
1	1	2	4	8
2	1	2	8	5
3	2	8	5	6
4	1	3	4	7
5	1	7	8	5
6	1	8	4	7